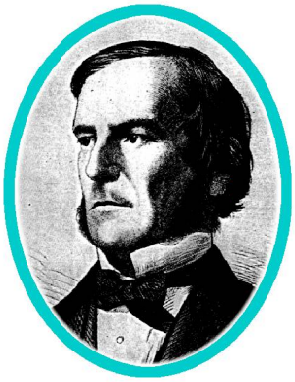
Chapter 14

MATHEMATICAL REASONING

❖ *There are few things which we know which are not capable of mathematical reasoning and when these can not, it is a sign that our knowledge of them is very small and confused and where a mathematical reasoning can be had, it is as great a folly to make use of another, as to grope for a thing in the dark when you have a candle stick standing by you. – ARTHENBOT* ❖

14.1 Introduction

In this Chapter, we shall discuss about some basic ideas of Mathematical Reasoning. All of us know that human beings evolved from the lower species over many millennia. The main asset that made humans "*superior*" to other species was the ability to reason. How well this ability can be used depends on each person's power of reasoning. How to develop this power? Here, we shall discuss the process of reasoning especially in the context of mathematics.



George Boole
(1815 - 1864)

In mathematical language, there are two kinds of reasoning – inductive and deductive. We have already discussed the inductive reasoning in the context of mathematical induction. In this Chapter, we shall discuss some fundamentals of deductive reasoning.

14.2 Statements

The basic unit involved in mathematical reasoning is a *mathematical statement*.

Let us start with two sentences:

In 2003, the president of India was a woman.

An elephant weighs more than a human being.

When we read these sentences, we immediately decide that the first sentence is false and the second is correct. There is no confusion regarding these. In mathematics such sentences are called *statements*.

On the other hand, consider the sentence:

Women are more intelligent than men.

Some people may think it is true while others may disagree. Regarding this sentence we cannot say whether it is always true or false . That means this sentence is ambiguous. Such a sentence is not acceptable as a statement in mathematics.

A sentence is called a mathematically acceptable statement if it is either true or false but not both. Whenever we mention a statement here, it is a "*mathematically acceptable*" statement.

While studying mathematics, we come across many such sentences. Some examples are:

Two plus two equals four.

The sum of two positive numbers is positive.

All prime numbers are odd numbers.

Of these sentences, the first two are *true* and the third one is *false*. There is no ambiguity regarding these sentences. Therefore, they are statements.

Can you think of an example of a sentence which is vague or ambiguous? Consider the sentence:

The sum of x and y is greater than 0

Here, we are not in a position to determine whether it is true or false, unless we know what x and y are. For example, it is false where $x = 1$, $y = -3$ and true when $x = 1$ and $y = 0$. Therefore, this sentence is not a statement. But the sentence:

For any natural numbers x and y, the sum of x and y is greater than 0

is a statement.

Now, consider the following sentences :

How beautiful!

Open the door.

Where are you going?

Are they statements? No, because the first one is an exclamation, the second an order and the third a question. None of these is considered as a statement in mathematical language. Sentences involving variable time such as "today", "tomorrow" or "yesterday" are not statements. This is because it is not known what time is referred here. For example, the sentence

Tomorrow is Friday

is not a statement. The sentence is correct (true) on a Thursday but not on other days. The same argument holds for sentences with pronouns unless a particular person is referred to and for variable places such as "here", "there" etc., For example, the sentences

She is a mathematics graduate.

Kashmir is far from here.

are not statements.

Here is another sentence

There are 40 days in a month.

Would you call this a statement? Note that the period mentioned in the sentence above is a "variable time" that is any of 12 months. But we know that the sentence is always false (irrespective of the month) since the maximum number of days in a month can never exceed 31. Therefore, this sentence is a statement. So, what makes a sentence a statement is the fact that the sentence is either true or false but not both.

While dealing with statements, we usually denote them by small letters *p, q, r,...* For example, we denote the statement "*Fire is always hot*" by *p*. This is also written as

p: Fire is always hot.

© BSEBPC WEB COPY-NOT TO BE PUBLISHED

Example 1 Check whether the following sentences are statements. Give reasons for your answer.

(i) 8 is less than 6.

(ii) Every set is a finite set.

(iii) The sun is a star.

(iv) Mathematics is fun.

(v) There is no rain without clouds.

(vi) How far is Chennai from here?

Solution (i) This sentence is false because 8 is greater than 6. Hence it is a statement.

(ii) This sentence is also false since there are sets which are not finite. Hence it is a statement.

(iii) It is a scientifically established fact that sun is a star and, therefore, this sentence is always true. Hence it is a statement.

(iv) This sentence is subjective in the sense that for those who like mathematics, it may be fun but for others it may not be. This means that this sentence is not always true. Hence it is not a statement.

(v) It is a scientifically established natural phenomenon that cloud is formed before it rains. Therefore, this sentence is always true. Hence it is a statement.

(vi) This is a question which also contains the word "Here". Hence it is not a statement.

The above examples show that whenever we say that a sentence is a statement we should always say why it is so. This "why" of it is more important than the answer.

EXERCISE 14.1

1. Which of the following sentences are statements? Give reasons for your answer.

(i) There are 35 days in a month.

(ii) Mathematics is difficult.

(iii) The sum of 5 and 7 is greater than 10.

(iv) The square of a number is an even number.

(v) The sides of a quadrilateral have equal length.

(vi) Answer this question.

(vii) The product of (–1) and 8 is 8.

(viii) The sum of all interior angles of a triangle is 180°.

(ix) Today is a windy day.

(x) All real numbers are complex numbers.

2. Give three examples of sentences which are not statements. Give reasons for the answers.

© DSTBPC WEB COPY NOT TO BE PUBLISHED

14.3 New Statements from Old

We now look into method for producing new statements from those that we already have. An English mathematician, "George Boole" discussed these methods in his book "The laws of Thought" in 1854. Here, we shall discuss two techniques.

As a first step in our study of statements, we look at an important technique that we may use in order to deepen our understanding of mathematical statements. This technique is to ask not only what it means to say that a given statement is true but also what it would mean to say that the given statement is not true.

14.3.1 *Negation of a statement* The denial of a statement is called the *negation* of the statement.

Let us consider the statement:

p: New Delhi is a city

The negation of this statement is

It is not the case that New Delhi is a city

This can also be written as

It is false that New Delhi is a city.

This can simply be expressed as

New Delhi is not a city.

Definition 1 If p is a statement, then the negation of p is also a statement and is denoted by $\sim p$, and read as 'not p'.

Note While forming the negation of a statement, phrases like, "It is not the case" or "It is false that" are also used.

Here is an example to illustrate how, by looking at the negation of a statement, we may improve our understanding of it.

Let us consider the statement

p: Everyone in Germany speaks German.

The denial of this sentence tells us that not everyone in Germany speaks German. This does not mean that no person in Germany speaks German. It says merely that at least one person in Germany does not speak German.

We shall consider more examples.

Example 2 Write the negation of the following statements.

(i) Both the diagonals of a rectangle have the same length.

(ii) $\sqrt{7}$ is rational.

Solution (i) This statement says that in a rectangle, both the diagonals have the same length. This means that if you take any rectangle, then both the diagonals have the same length. The negation of this statement is

It is false that both the diagonals in a rectangle have the same length

This means the statement

There is atleast one rectangle whose both diagonals do not have the same length.

(ii) The negation of the statement in (ii) may also be written as

It is not the case that $\sqrt{7}$ *is rational.*

This can also be rewritten as

$\sqrt{7}$ *is not rational.*

© NCERT NOT TO BE REPUBLISHED WEB COPY

Example 3 Write the negation of the following statements and check whether the resulting statements are true,

(i) Australia is a continent.

(ii) There does not exist a quadrilateral which has all its sides equal.

(iii) Every natural number is greater than 0.

(iv) The sum of 3 and 4 is 9.

Solution (i) The negation of the statement is

It is false that Australia is a continent.

This can also be rewritten as

Australia is not a continent.

We know that this statement is false.

(ii) The negation of the statement is

It is not the case that there does not exist a quadrilateral which has all its sides equal.

This also means the following:

There exists a quadrilateral which has all its sides equal.

This statement is true because we know that square is a quadrilateral such that its four sides are equal.

(iii) The negation of the statement is

It is false that every natural number is greater than 0.

This can be rewritten as

There exists a natural number which is not greater than 0.

This is a false statement.

(iv) The negation is

It is false that the sum of 3 *and* 4 *is* 9.

This can be written as

The sum of 3 and 4 is not equal to 9.

This statement is true.

14.3.2 *Compound statements* Many mathematical statements are obtained by combining one or more statements using some connecting words like “and”, “or”, etc. Consider the following statement

p: There is something wrong with the bulb or with the wiring.

This statement tells us that there is something wrong with the bulb or there is

© BSTBPC WEB COPY NOT TO BE PUBLISHED

something wrong with the wiring. That means the given statement is actually made up of two smaller statements:

q: There is something wrong with the bulb.

r: There is something wrong with the wiring.

connected by "or"

Now, suppose two statements are given as below:

p: 7 is an odd number.

q: 7 is a prime number.

These two statements can be combined with "and"

r: 7 is both odd and prime number.

This is a compound statement.

This leads us to the following definition:

Definition 2 **A Compound Statement** is a statement which is made up of two or more statements. In this case, each statement is called a ***component statement.***

Let us consider some examples.

Example 4 Find the component statements of the following compound statements.

(i) The sky is blue and the grass is green.

(ii) It is raining and it is cold.

(iii) All rational numbers are real and all real numbers are complex.

(iv) 0 is a positive number or a negative number.

Solution Let us consider one by one

(i) The component statements are

p: The sky is blue.

q: The grass is green.

The connecting word is 'and'.

(ii) The component statements are

p: It is raining.

q: It is cold.

The connecting word is 'and'.

(iii) The component statements are

p: All rational numbers are real.

q: All real numbers are complex.

The connecting word is 'and'.

(iv) The component statements are

© RSTBPC WEB COPY-NOT TO BE PUBLISHED

p: 0 is a positive number.

q: 0 is a negative number.

The connecting word is 'or'.

Example 5 Find the component statements of the following and check whether they are true or not.

(i) A square is a quadrilateral and its four sides equal.

(ii) All prime numbers are either even or odd.

(iii) A person who has taken Mathematics or Computer Science can go for MCA.

(iv) Chandigarh is the capital of Haryana and UP.

(v) $\sqrt{2}$ is a rational number or an irrational number.

(vi) 24 is a multiple of 2, 4 and 8.

Solution (i) The component statements are

p: A square is a quadrilateral.

q: A square has all its sides equal.

We know that both these statements are true. Here the connecting word is 'and'.

(ii) The component statements are

p: All prime numbers are odd numbers.

q: All prime numbers are even numbers.

Both these statements are false and the connecting word is 'or'.

(iii) The component statements are

p: A person who has taken Mathematics can go for MCA.

q: A person who has taken computer science can go for MCA.

Both these statements are true. Here the connecting word is 'or'.

(iv) The component statements are

p: Chandigarh is the capital of Haryana.

q: Chandigarh is the capital of UP.

The first statement is true but the second is false. Here the connecting word is 'and'.

(v) The component statements are

© DSTBPC WEB COPY NOT TO BE PUBLISHED

p: $\sqrt{2}$ *is a rational number.*

q: $\sqrt{2}$ *is an irrational number.*

The first statement is false and second is true. Here the connecting word is 'or'.

(vi) The component statements are

p: 24 *is a multiple of* 2.

q: 24 *is a multiple of* 4.

r: 24 *is a multiple of* 8.

All the three statements are true. Here the connecting words are 'and'.

Thus, we observe that compound statements are actually made-up of two or more statements connected by the words like "and", "or", etc. These words have special meaning in mathematics. We shall discuss this mattter in the following section.

EXERCISE 14.2

1. Write the negation of the following statements:

(i) Chennai is the capital of Tamil Nadu.

(ii) $\sqrt{2}$ is not a complex number

(iii) All triangles are not equilateral triangle.

(iv) The number 2 is greater than 7.

(v) Every natural number is an integer.

2. Are the following pairs of statements negations of each other:

(i) The number x is not a rational number.
The number x is not an irrational number.

(ii) The number x is a rational number.
The number x is an irrational number.

3. Find the component statements of the following compound statements and check whether they are true or false.

(i) Number 3 is prime or it is odd.

(ii) All integers are positive or negative.

(iii) 100 is divisible by 3, 11 and 5.

14.4 Special Words/Phrases

Some of the connecting words which are found in compound statements like **"And",**

© NCERT NOT TO BE PUBLISHED

"Or", etc. are often used in Mathematical Statements. These are called connectives. When we use these compound statements, it is necessary to understand the role of these words. We discuss this below.

14.4.1 ***The word "And"*** Let us look at a compound statement with "And".

p: A point occupies a position and its location can be determined.

The statement can be broken into two component statements as

q: A point occupies a position.

r: Its location can be determined.

Here, we observe that both statements are true.

Let us look at another statement.

p: 42 *is divisible by* 5, 6 *and* 7.

This statement has following component statements

q: 42 is divisible by 5.

r: 42 is divisible by 6.

s: 42 is divisible by 7.

Here, we know that the first is false while the other two are true.

We have the following rules regarding the connective "And"

1. The compound statement with 'And' is true if all its component statements are true.
2. The component statement with 'And' is false if any of its component statements is false (this includes the case that some of its component statements are false or all of its component statements are false).

© NCERT NOT TO BE PUBLISHED

Example 6 Write the component statements of the following compound statements and check whether the compound statement is true or false.

(i) A line is straight and extends indefinitely in both directions.

(ii) 0 is less than every positive integer and every negative integer.

(iii) All living things have two legs and two eyes.

Solution (i) The component statements are

p: A line is straight.

q: A line extends indefinitely in both directions.

Both these statements are true, therefore, the compound statement is true.

(ii) The component statements are

p: 0 is less than every positive integer.

q: 0 is less than every negative integer.

The second statement is false. Therefore, the compound statement is false.

(iii) The two component statements are

p: All living things have two legs.

q: All living things have two eyes.

Both these statements are false. Therefore, the compound statement is false.

Now, consider the following statement.

p: A mixture of alcohol and water can be separated by chemical methods.

This sentence cannot be considered as a compound statement with "And". Here the word "And" refers to two things – alcohol and water.

This leads us to an important note.

Note Do not think that a statement with "And" is always a compound statement as shown in the above example. Therefore, the word "And" is not used as a connective.

14.4.2 *The word "Or"* Let us look at the following statement.

p: Two lines in a plane either intersect at one point or they are parallel.

We know that this is a true statement. What does this mean? This means that if two lines in a plane intersect, then they are not parallel. Alternatively, if the two lines are not parallel, then they intersect at a point. That is this statement is true in both the situations.

In order to understand statements with "Or" we first notice that the word "Or" is used in two ways in English language. Let us first look at the following statement.

p: An ice cream or pepsi is available with a Thali in a restaurant.

This means that a person who does not want ice cream can have a pepsi along with *Thali* or one does not want pepsi can have an ice cream along with *Thali*. That is, who do not want a pepsi can have an ice cream. A person cannot have both ice cream and pepsi. This is called an ***exclusive "Or".***

Here is another statement.

A student who has taken biology or chemistry can apply for M.Sc. microbiology programme.

Here we mean that the students who have taken both biology and chemistry can apply for the microbiology programme, as well as the students who have taken only one of these subjects. In this case, we are using **inclusive "Or".**

It is important to note the difference between these two ways because we require this when we check whether the statement is true or not.

Let us look at an example.

Example 7 For each of the following statements, determine whether an **inclusive "Or"** or **exclusive "Or"** is used. Give reasons for your answer.

(i) To enter a country, you need a passport or a voter registration card.

(ii) The school is closed if it is a holiday or a Sunday.

(iii) Two lines intersect at a point or are parallel.

(iv) Students can take French or Sanskrit as their third language.

Solution (i) Here "Or" is inclusive since a person can have both a passport and a voter registration card to enter a country.

(ii) Here also "Or" is inclusive since school is closed on holiday as well as on Sunday.

(iii) Here "Or" is exclusive because it is not possible for two lines to intersect and parallel together.

(iv) Here also "Or" is exclusive because a student cannot take both French and Sanskrit.

Rule for the compound statement with 'Or'

1. A compound statement with an 'Or' is true when one component statement is true or both the component statements are true.
2. A compound statement with an 'Or' is false when both the component statements are false.

For example, consider the following statement.

p: Two lines intersect at a point or they are parallel

The component statements are

q: *Two lines intersect at a point.*

r: *Two lines are parallel.*

Then, when *q* is true *r* is false and when *r* is true *q* is false. Therefore, the compound statement *p* is true.

Consider another statement.

p: 125 *is a multiple of* 7 *or* 8.

Its component statements are

q: 125 *is a multiple of* 7.

r: 125 *is a multiple of* 8.

Both *q* and *r* are false. Therefore, the compound statement *p* is false.

Again, consider the following statement:

p: The school is closed, if there is a holiday or Sunday.

The component statements are

q: *School is closed if there is a holiday.*

r: *School is closed if there is a Sunday.*

Both *q* and *r* are true, therefore, the compound statement is true.
Consider another statement.

p: *Mumbai is the capital of Kolkata or Karnataka.*

The component statements are

q: *Mumbai is the capital of Kolkata.*

r: *Mumbai is the capital of Karnataka.*

Both these statements are false. Therefore, the compound statement is false.
Let us consider some examples.

Example 8 Identify the type of "Or" used in the following statements and check whether the statements are true or false:

(i) $\sqrt{2}$ is a rational number or an irrational number.

(ii) To enter into a public library children need an identity card from the school or a letter from the school authorities.

(iii) A rectangle is a quadrilateral or a 5-sided polygon.

Solution (i) The component statements are

p: $\sqrt{2}$ *is a rational number.*

q: $\sqrt{2}$ *is an irrational number.*

Here, we know that the first statement is false and the second is true and "Or" is exclusive. Therefore, the compound statement is true.

(ii) The component statements are

p: To get into a public library children need an identity card.

q: To get into a public library children need a letter from the school authorities.

Children can enter the library if they have either of the two, an identity card or the letter, as well as when they have both. Therefore, it is inclusive "Or" the compound statement is also true when children have both the card and the letter.

(iii) Here "Or" is exclusive. When we look at the component statements, we get that the statement is true.

© RS IBPC WEB COPY NOT TO BE REPUBLISHED

14.4.3 *Quantifiers* Quantifiers are phrases like, "There exists" and "For all". Another phrase which appears in mathematical statements is "there exists". For example, consider the statement. *p*: *There exists a rectangle whose all sides are equal*. This means that there is atleast one rectangle whose all sides are equal.

A word closely connected with "there exists" is "for every" (or for all). Consider a statement.

p: For every prime number p, $\sqrt{p}$ *is an irrational number.*

This means that if S denotes the set of all prime numbers, then for all the members *p* of the set S, $\sqrt{p}$ is an irrational number.

In general, a mathematical statement that says "for every" can be interpreted as saying that all the members of the given set S where the property applies must satisfy that property.

We should also observe that it is important to know precisely where in the sentence a given connecting word is introduced. For example, compare the following two sentences:

1. For every positive number x there exists a positive number y such that $y < x$.
2. There exists a positive number y such that for every positive number x, we have $y < x$.

Although these statements may look similar, they do not say the same thing. As a matter of fact, (1) is true and (2) is false. Thus, in order for a piece of mathematical writing to make sense, all of the symbols must be carefully introduced and each symbol must be introduced precisely at the right place – not too early and not too late.

The words "And" and "Or" are called *connectives* and "There exists" and "For all" are called *quantifiers*.

Thus, we have seen that many mathematical statements contain some special words and it is important to know the meaning attached to them, especially when we have to check the validity of different statements.

EXERCISE 14.3

1. For each of the following compound statements first identify the connecting words and then break it into component statements.

(i) All rational numbers are real and all real numbers are not complex.
(ii) Square of an integer is positive or negative.
(iii) The sand heats up quickly in the Sun and does not cool down fast at night.
(iv) $x = 2$ and $x = 3$ are the roots of the equation $3x^2 - x - 10 = 0$.

© BSTBPC NOT TO BE PUBLISHED

2. Identify the quantifier in the following statements and write the negation of the statements.
   (i) There exists a number which is equal to its square.
   (ii) For every real number x, x is less than $x + 1$.
   (iii) There exists a capital for every state in India.
3. Check whether the following pair of statements are negation of each other. Give reasons for your answer.
   (i) $x + y = y + x$ is true for every real numbers x and y.
   (ii) There exists real numbers x and y for which $x + y = y + x$.
4. State whether the "Or" used in the following statements is "exclusive "or" inclusive. Give reasons for your answer.
   (i) Sun rises or Moon sets.
   (ii) To apply for a driving licence, you should have a ration card or a passport.
   (iii) All integers are positive or negative.

14.5 Implications

In this Section, we shall discuss the implications of "if-then", "only if" and "if and only if".

The statements with "if-then" are very common in mathematics. For example, consider the statement.

r: If you are born in some country, then you are a citizen of that country.

When we look at this statement, we observe that it corresponds to two statements p and q given by

p : you are born in some country.
q : you are citizen of that country.

Then the sentence "if p then q" says that in the event if p is true, then q must be true.

One of the most important facts about the sentence "if p then q" is that it does not say any thing (or places no demand) on q when p is false. For example, if you are not born in the country, then you cannot say anything about q. To put it in other words" not happening of p has no effect on happening of q.

Another point to be noted for the statement "if p then q" is that the statement does not imply that p happens.

There are several ways of understanding "if p then q" statements. We shall illustrate these ways in the context of the following statement.

r: If a number is a multiple of 9, then it is a multiple of 3.

Let p and q denote the statements

p : a number is a multiple of 9.
q: a number is a multiple of 3.

© BSTBPC WEB COPY NOT TO BE PUBLISHED

Then, if p then q is the same as the following:

1. p **implies** q is denoted by $p \Rightarrow q$. The symbol $\Rightarrow$ stands for implies.
   This says that a number is a multiple of 9 implies that it is a multiple of 3.
2. p is a sufficient condition for q.
   This says that knowing that a number as a multiple of 9 is sufficient to conclude that it is a multiple of 3.
3. p only if q.
   This says that a number is a multiple of 9 only if it is a multiple of 3.
4. q is a necessary condition for p.
   This says that when a number is a multiple of 9, it is necessarily a multiple of 3.
5. $\sim q$ implies $\sim p$.
   This says that if a number is not a multiple of 3, then it is not a multiple of 9.

14.5.1 ***Contrapositive and converse*** Contrapositive and converse are certain other statements which can be formed from a given statement with "if-then".

For example, let us consider the following "if-then" statement.

If the physical environment changes, then the biological environment changes.

Then the contrapositive of this statement is

If the biological environment does not change, then the physical environment does not change.

Note that both these statements convey the same meaning.

To understand this, let us consider more examples.

Example 9 Write the contrapositive of the following statement:

(i) If a number is divisible by 9, then it is divisible by 3.

(ii) If you are born in India, then you are a citizen of India.

(iii) If a triangle is equilateral, it is isosceles.

Solution The contrapositive of the these statements are

(i) If a number is not divisible by 3, it is not divisible by 9.

(ii) If you are not a citizen of India, then you were not born in India.

(iii) If a triangle is not isosceles, then it is not equilateral.

The above examples show the contrapositive of the statement if p, then q is "if $\sim q$, then $\sim p$".

Next, we shall consider another term called *converse*.

The converse of a given statement "if p, then q" is if q, then p.

© NCERT NOT TO BE REPUBLISHED

For example, the converse of the statement

p: If a number is divisible by 10, *it is divisible by* 5 *is*

q: If a number is divisible by 5, *then it is divisible by* 10.

Example 10 Write the converse of the following statements.

(i) If a number n is even, then n^2 is even.
(ii) If you do all the exercises in the book, you get an A grade in the class.
(iii) If two integers a and b are such that $a > b$, then $a - b$ is always a positive integer.

Solution The converse of these statements are

(i) If a number n^2 is even, then n is even.
(ii) If you get an A grade in the class, then you have done all the exercises of the book.
(iii) If two integers a and b are such that $a - b$ is always a positive integer, then $a > b$.

Let us consider some more examples.

Example 11 For each of the following compound statements, first identify the corresponding component statements. Then check whether the statements are true or not.

(i) If a triangle ABC is equilateral, then it is isosceles.
(ii) If a and b are integers, then ab is a rational number.

Solution (i) The component statements are given by

p : Triangle ABC is equilateral.

q : Triangle ABC is Isosceles.

Since an equilateral triangle is isosceles, we infer that the given compound statement is true.

(ii) The component statements are given by

p : a and *b* are integers.

q : ab is a rational number.

since the product of two integers is an integer and therefore a rational number, the compound statement is true.

'If and only if', represented by the symbol '⇔' means the following equivalent forms for the given statements p and q.

(i) p if and only if q
(ii) q if and only if p

© NCERT NOT TO BE PUBLISHED
WEB COPY

(iii) p is necessary and sufficient condition for q and vice-versa

(iv) $p \Leftrightarrow q$

Consider an example.

Example 12 Given below are two pairs of statements. Combine these two statements using "if and only if ".

(i) *p:* If a rectangle is a square, then all its four sides are equal.
q: If all the four sides of a rectangle are equal, then the rectangle is a square.

(ii) *p:* If the sum of digits of a number is divisible by 3, then the number is divisible by 3.

q: If a number is divisible by 3, then the sum of its digits is divisible by 3.

Solution (i) A rectangle is a square if and only if all its four sides are equal.

(ii) A number is divisible by 3 if and only if the sum of its digits is divisible by 3.

EXERCISE 14.4

1. Rewrite the following statement with "if-then" in five different ways conveying the same meaning.

If a natural number is odd, then its square is also odd.

2. Write the contrapositive and converse of the following statements.

(i) If x is a prime number, then x is odd.

(ii) If the two lines are parallel, then they do not intersect in the same plane.

(iii) Something is cold implies that it has low temperature.

(iv) You cannot comprehend geometry if you do not know how to reason deductively.

(v) x is an even number implies that x is divisible by 4.

3. Write each of the following statements in the form "if-then"

(i) You get a job implies that your credentials are good.

(ii) The Bannana trees will bloom if it stays warm for a month.

(iii) A quadrilateral is a parallelogram if its diagonals bisect each other.

(iv) To get an A^+ in the class, it is necessary that you do all the exercises of the book.

© NCERT NOT TO BE PUBLISHED

4. Given statements in (a) and (b). Identify the statements given below as contrapositive or converse of each other.

(a) If you live in Delhi, then you have winter clothes.

(i) If you do not have winter clothes, then you do not live in Delhi.

(ii) If you have winter clothes, then you live in Delhi.

(b) If a quadrilateral is a parallelogram, then its diagonals bisect each other.

(i) If the diagonals of a quadrilateral do not bisect each other, then the quadrilateral is not a parallelogram.

(ii) If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

14.6 Validating Statements

In this Section, we will discuss when a statement is true. To answer this question, one must answer all the following questions.

What does the statement mean? What would it mean to say that this statement is true and when this statement is not true?

The answer to these questions depend upon which of the special words and phrases "and", "or", and which of the implications "if and only", "if-then", and which of the quantifiers "for every", "there exists", appear in the given statement.

Here, we shall discuss some techniques to find when a statement is valid.

We shall list some general rules for checking whether a statement is true or not.

Rule 1 ***If p and q are mathematical statements, then in order to show that the statement "p and q" is true, the following steps are followed.***

Step-1 Show that the statement p is true.

Step-2 Show that the statement q is true.

Rule 2 ***Statements with "Or"***

If p and q are mathematical statements , then in order to show that the statement "p or q" is true, one must consider the following.

Case 1 By assuming that p is false, show that q must be true.

Case 2 By assuming that q is false, show that p must be true.

Rule 3 ***Statements with "If-then"***

© NCERT NOT TO BE PUBLISHED

In order to prove the statement "if p then q" we need to show that *any one* of the following case is true.

Case 1 By assuming that p is true, prove that q must be true.(Direct method)

Case 2 By assuming that q is false, prove that p must be false.(Contrapositive method)

Rule 4 ***Statements with "if and only if "***

In order to prove the statement "p if and only if q", we need to show.

(i) *If p is true, then q is true* and (ii) *If q is true, then p is true*

Now we consider some examples.

Example 13 Check whether the following statement is true or not.
If $x, y \in \mathbf{Z}$ are such that x and y are odd, then xy is odd.

Solution Let $p : x, y \in \mathbf{Z}$ such that x and y are odd

$$q : xy \text{ is odd}$$

To check the validity of the given statement, we apply Case 1 of Rule 3. That is assume that if p is true, then q is true.
p is true means that x and y are odd integers. Then

$x = 2m + 1$, for some integer m. $y = 2n + 1$, for some integer n. Thus

$$\begin{aligned} xy &= (2m + 1)(2n + 1) \\ &= 2(2mn + m + n) + 1 \end{aligned}$$

This shows that xy is odd. Therefore, the given statement is true.

Suppose we want to check this by using Case 2 of Rule 3, then we will proceed as follows.

We assume that q is not true. This implies that we need to consider the negation of the statement q. This gives the statement

$$\sim q : \text{Product } xy \text{ is even.}$$

This is possible only if either x or y is even. This shows that p is not true. Thus we have shown that

$$\sim q \Rightarrow \sim p$$

Note The above example illustrates that to prove $p \Rightarrow q$, it is enough to show $\sim q \Rightarrow \sim p$ which is the contrapositive of the statement $p \Rightarrow q$.

Example 14 Check whether the following statement is true or false by proving its contrapositive. If $x, y \in \mathbf{Z}$ such that xy is odd, then both x and y are odd.

Solution Let us name the statements as below

© NCERT not to be republished
WEB COPY NOT TO BE PUBLISHED

p : xy is odd.

q : both x and y are odd.

We have to check whether the statement $p \Rightarrow q$ is true or not, that is, by checking its contrapositive statement i.e., $\sim q \Rightarrow \sim p$

Now $\sim q$: It is false that both x and y are odd. This implies that x (or y) is even.

Then $x = 2n$ for some integer n.

Therefore, $xy = 2ny$ for some integer n. This shows that xy is even. That is $\sim p$ is true. Thus, we have shown that $\sim q \Rightarrow \sim p$ and hence the given statement is true.

Now what happens when we combine an implication and its converse? Next, we shall discuss this.

Let us consider the following statements.

p : A tumbler is half empty.

q : A tumbler is half full.

We know that if the first statement happens, then the second happens and also if the second happens, then the first happens. We can express this fact as

If a tumbler is half empty, then it is half full.

If a tumbler is half full, then it is half empty.

We combine these two statements and get the following:

A tumbler is half empty if and only if it is half full.

Now, we discuss another method.

14.6.1 *By Contradiction* Here to check whether a statement p is true, we assume that p is not true i.e. $\sim p$ is true. Then, we arrive at some result which contradicts our assumption. Therefore, we conclude that p is true.

Example 15 Verify by the method of contradiction.

p: $\sqrt{7}$ is irrational

Solution In this method, we assume that the given statement is false. That is we assume that $\sqrt{7}$ is rational. This means that there exists positive integers a and b such that $\sqrt{7} = \frac{a}{b}$, where a and b have no common factors. Squaring the equation,

© NCERT NOT TO BE PUBLISHED

we get $7 = \frac{a^2}{b^2} \Rightarrow a^2 = 7b^2 \Rightarrow$ 7 divides a. Therefore, there exists an integer c such that $a = 7c$. Then $a^2 = 49c^2$ and $a^2 = 7b^2$

Hence, $7b^2 = 49c^2 \Rightarrow b^2 = 7c^2 \Rightarrow$ 7 divides b. But we have already shown that 7 divides a. This implies that 7 is a common factor of both of a and b which contradicts our earlier assumption that a and b have no common factors. This shows that the assumption $\sqrt{7}$ is rational is wrong. Hence, the statement $\sqrt{7}$ is irrational is true.

Next, we shall discuss a method by which we may show that a statement is false. The method involves giving an ***example of a situation where the statement is not valid.*** Such an example is called a ***counter example***. The name itself suggests that this is an example to counter the given statement.

Example 16 By giving a counter example, show that the following statement is false. If n is an odd integer, then n is prime.

Solution The given statement is in the form "if p then q" we have to show that this is false. For this purpose we need to show that if p then $\sim q$. To show this we look for an odd integer n which is not a prime number. 9 is one such number. So $n = 9$ is a counter example. Thus, we conclude that the given statement is false.

In the above, we have discussed some techniques for checking whether a statement is true or not.

Note In mathematics, counter examples are used to disprove the statement. However, generating examples in favour of a statement do not provide validity of the statement.

EXERCISE 14.5

1. Show that the statement
   p: "If x is a real number such that $x^3 + 4x = 0$, then x is 0" is true by
   (i) direct method, (ii) method of contradiction, (iii) method of contrapositive
2. Show that the statement "For any real numbers a and b, $a^2 = b^2$ implies that $a = b$" is not true by giving a counter-example.
3. Show that the following statement is true by the method of contrapositive.
   p: *If x is an integer and x^2 is even, then x is also even.*
4. By giving a counter example, show that the following statements are not true.
   (i) *p*: If all the angles of a triangle are equal, then the triangle is an obtuse angled triangle.
   (ii) *q*: The equation $x^2 - 1 = 0$ does not have a root lying between 0 and 2.

5. Which of the following statements are true and which are false? In each case give a valid reason for saying so.

(i) *p:* Each radius of a circle is a chord of the circle.

(ii) *q:* The centre of a circle bisects each chord of the circle.

(iii) *r:* Circle is a particular case of an ellipse.

(iv) *s:* If x and y are integers such that $x > y$, then $-x < -y$.

(v) *t :* $\sqrt{11}$ is a rational number.

Miscellaneous Examples

Example 17 Check whether "Or" used in the following compound statement is exclusive or inclusive? Write the component statements of the compound statements and use them to check whether the compound statement is true or not. Justify your answer.

t: you are wet when it rains or you are in a river.

Solution "Or" used in the given statement is inclusive because it is possible that it rains and you are in the river.

The component statements of the given statement are

p : you are wet when it rains.

q : You are wet when you are in a river.

Here both the component statements are true and therefore, the compound statement is true.

Example 18 Write the negation of the following statements:

(i) *p:* For every real number x, $x^2 > x$.

(ii) *q:* There exists a rational number x such that $x^2 = 2$.

(iii) *r:* All birds have wings.

(iv) *s:* All students study mathematics at the elementary level.

Solution (i) The negation of p is "It is false that p is" which means that the condition $x^2 > x$ does not hold for all real numbers. This can be expressed as

$\sim p$: There exists a real number x such that $x^2 < x$.

(ii) Negation of q is "it is false that q", Thus $\sim q$ is the statement.

$\sim q$: There does not exist a rational number x such that $x^2 = 2$.

This statement can be rewritten as

$\sim q$: For all real numbers x, $x^2 \neq 2$

(iii) The negation of the statement is

$\sim r$: There exists a bird which have no wings.

© BSTBPC WEB COPY NOT TO BE PUBLISHED

(iv) The negation of the given statement is $\sim s$: There exists a student who does not study mathematics at the elementary level.

Example 19 Using the words "necessary and sufficient" rewrite the statement "The integer n is odd if and only if n^2 is odd". Also check whether the statement is true.

Solution The necessary and sufficient condition that the integer n be odd is n^2 must be odd. Let p and q denote the statements

p : the integer n is odd.

q : n^2 is odd.

To check the validity of "p if and only if q", we have to check whether "if p then q" and "if q then p" is true.

Case 1 If p, then q

If p, then q is the statement:

If the integer n is odd, then n^2 is odd. We have to check whether this statement is true. Let us assume that n is odd. Then $n = 2k + 1$ when k is an integer. Thus

$$n^2 = (2k + 1)^2$$

$$= 4k^2 + 4k + 1$$

Therefore, n^2 is one more than an even number and hence is odd.

Case 2 If q, then p

If q, then p is the statement

If n is an integer and n^2 is odd, then n is odd.

We have to check whether this statement is true. We check this by contrapositive method. The contrapositive of the given statement is:

If n is an even integer, then n^2 is an even integer

n is even implies that $n = 2k$ for some k. Then $n^2 = 4k^2$. Therefore, n^2 is even.

Example 20 For the given statements identify the necessary and sufficient conditions.
t: If you drive over 80 km per hour, then you will get a fine.

Solution Let p and q denote the statements:

p : you drive over 80 km per hour.

q : you will get a fine.

The implication if p, then q indicates that p is sufficient for q. That is driving over 80 km per hour is sufficient to get a fine.
Here the sufficient condition is "driving over 80 km per hour":
Similarly, if p, then q also indicates that q is necessary for p. That is

© BSTBPC WEB COPY NOT TO BE PUBLISHED

When you drive over 80 km per hour, you will necessarily get a fine.
Here the necessary condition is "getting a fine".

Miscellaneous Exercise on Chapter 14

1. Write the negation of the following statements:

(i) *p:* For every positive real number x, the number $x - 1$ is also positive.
(ii) *q:* All cats scratch.
(iii) *r:* For every real number x, either $x > 1$ or $x < 1$.
(iv) *s:* There exists a number x such that $0 < x < 1$.

2. State the converse and contrapositive of each of the following statements:

(i) *p:* A positive integer is prime only if it has no divisors other than 1 and itself.
(ii) *q:* I go to a beach whenever it is a sunny day.
(iii) *r:* If it is hot outside, then you feel thirsty.

3. Write each of the statements in the form "if p, then q"

(i) *p:* It is necessary to have a password to log on to the server.
(ii) *q:* There is traffic jam whenever it rains.
(iii) *r:* You can access the website only if you pay a subsciption fee.

4. Rewrite each of the following statements in the form "p if and only if q"

(i) *p:* If you watch television, then your mind is free and if your mind is free, then you watch television.
(ii) *q:* For you to get an A grade, it is necessary and sufficient that you do all the homework regularly.
(iii) *r:* If a quadrilateral is equiangular, then it is a rectangle and if a quadrilateral is a rectangle, then it is equiangular.

5. Given below are two statements

p : 25 is a multiple of 5.

q : 25 is a multiple of 8.

Write the compound statements connecting these two statements with "And" and "Or". In both cases check the validity of the compound statement.

6. Check the validity of the statements given below by the method given against it.

(i) *p:* The sum of an irrational number and a rational number is irrational (by contradiction method).
(ii) *q:* If n is a real number with $n > 3$, then $n^2 > 9$ (by contradiction method).

7. Write the following statement in five different ways, conveying the same meaning.
p: If a triangle is equiangular, then it is an obtuse angled triangle.

© NCERT NOT TO BE REPUBLISHED

Summary

- A mathematically acceptable statement is a sentence which is either true or false.
- Explained the terms:
  - Negation of a statement *p:* If p denote a statement, then the negation of p is denoted by $\sim p$.
  - Compound statements and their related component statements:
    A statement is a compound statement if it is made up of two or more smaller statements. The smaller statements are called component statements of the compound statement.
  - The role of "And", "Or", "There exists" and "For every" in compound statements.
  - The meaning of implications "If ", "only if ", " if and only if ".
    A sentence with if p, then q can be written in the following ways.
  - p implies q (denoted by $p \Rightarrow q$)
  - p is a sufficient condition for q
  - q is a necessary condition for p
  - p only if q
  - $\sim q$ implies $\sim p$
  - The contrapositive of a statement $p \Rightarrow q$ is the statement $\sim q \Rightarrow \sim p$. The converse of a statement $p \Rightarrow q$ is the statement $q \Rightarrow p$.
    $p \Rightarrow q$ together with its converse, gives p if and only if q.
- The following methods are used to check the validity of statements:
  (i) direct method
  (ii) contrapositive method
  (iii) method of contradiction
  (iv) using a counter example.

Historical Note

The first treatise on logic was written by *Aristotle* (384 B.C.-322 B.C.). It was a collection of rules for deductive reasoning which would serve as a basis for the study of every branch of knowledge. Later, in the seventeenth century, German mathematician G. W. Leibnitz (1646 – 1716) conceived the idea of using symbols in logic to mechanise the process of deductive reasoning. His idea was realised in the nineteenth century by the English mathematician *George Boole* (1815–1864) and *Augustus De Morgan* (1806–1871) , who founded the modern subject of symbolic logic.